L20 March 3 Cpt

Wednesday, February 25, 2015 8:53 PM

Theorem If each X_β is compact then

$$\prod_{\alpha\in I} X_\alpha \text{ is also compact}$$

* I is infinite, **Tychonoff Theorem**
* I is finite, proved below.

Let both X, Y be compact and

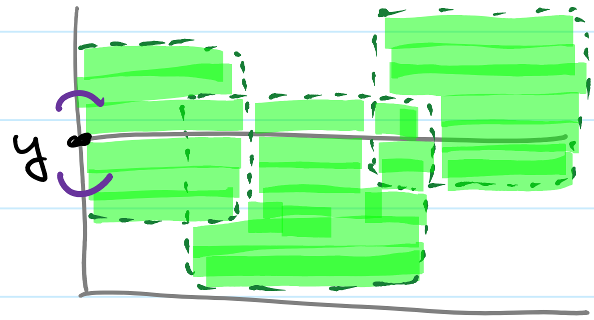
$$\mathcal{G} \subset \mathcal{J}_{X\times Y} \quad \text{with} \quad \bigcup\mathcal{G} = X\times Y$$

For simplicity, assume all sets in $\mathcal{G}$ are of the form $U\times V$ with $U\in\mathcal{J}_X$, $V\in\mathcal{J}_Y$

For each fixed $y\in Y$, $\bigcup\mathcal{G} \supset X\times\{y\}$

$\Sigma_y = \{U_k\times V_k : k=1,\dots,n\} \subset \mathcal{G}$

$\bigcup_{k=1}^{n}(U_k\times V_k) \supset X\times\{y\}$



$\therefore\ y\in V_k$ for each k

$$y\in V_y = \bigcap_{k=1}^{n} V_k$$

Do this for each y, $\{\bigcup V_y : y\in Y\} \supset Y$

$\exists\ \{V_{y_1}, V_{y_2}, \dots, V_{y_m}\}$ such that $\bigcup_{l=1}^{m} V_{y_l} \supset Y$

Then, $\bigcup_{l=1}^{m} \Sigma_{y_l} \subset \mathcal{G}$ is a finite subcover for $X\times Y$

For a general $\mathcal{C} \subset \mathcal{T}_{X\times Y}$

finite $\mathcal{E} \subset \mathcal{C}$

$\mathcal{C}^* \subset \mathcal{T}_{X\times Y}$ of the form $U\times V$

$\mathcal{E}^*$ finite subcover

Qu. Is the following correct?

X is compact $\Longleftrightarrow$ $\forall\, \mathcal{G} \subset \mathcal{B}$, a base, with $\bigcup \mathcal{G} = X$, $\exists$ finite $\mathcal{E} \subset \mathcal{G}$, $\bigcup \mathcal{E} = X$

For arbitrary product It is easier to consider intersection of closed sets

1. $\sim\left(\bigcup \mathcal{G} \supset X\right) \Longleftrightarrow X \setminus \bigcup \mathcal{G} \neq \emptyset$

$$X \setminus \bigcup \mathcal{G} = \bigcap \{\underbrace{X \setminus C}_{\text{closed}} : C \in \mathcal{G}\}, \quad \mathcal{G} \subset \mathcal{T}$$

2. If $\mathcal{G} \subset \mathcal{T}$ with $\bigcup \mathcal{G} = X$ then $\exists$ finite $\mathcal{E} \subset \mathcal{G}$ with $\bigcup \mathcal{E} = X$

negation $\forall$ finite $\mathcal{E} \subset \mathcal{G}$, $\bigcap \{X \setminus E : E \in \mathcal{E}\} \neq \emptyset$

3. Contrapositive

If ( ) then $\bigcap \{X \setminus G : G \in \mathcal{G}\} \neq \emptyset$

L20 p3

Monday, March 2, 2015 3:11 PM

X is compact $\iff$

$\forall$ family $\mathcal{H}$ of closed sets in X

if $\forall$ finite $\mathcal{F} \subset \mathcal{H}$, $\cap \mathcal{F} \neq \phi$ then $\cap \mathcal{H} \neq \phi$

$\iff$ $\forall$ family $\mathcal{H}$ of sets in X

if $\forall$ finite $\mathcal{F} \subset \mathcal{H}$, $\cap \overline{\mathcal{F}} \neq \phi$ then $\cap \overline{\mathcal{H}} \neq \phi$

$\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$

finite closure intersection property

Zorn's Lemma

$\forall$ maximal family $\mathcal{M}$ of sets in X with the f.c.i.p., we have $\cap \overline{\mathcal{M}} \neq \phi$

Good property of maximality of $\mathcal{M}$

(1) It is closed under finite intersection

(2) If $A \subset X$ satisfies $A \cap M \neq \phi$ $\forall M \in \mathcal{M}$ then $A \in \mathcal{M}$

(1) & (2) helps us that

if $x \in \prod X_\alpha$ satisfies $x_\beta \in \overline{\pi_\beta(M)}$ $\forall \beta$ $\forall M \in \mathcal{M}$

then $x \in \overline{M}$ $\forall M \in \mathcal{M}$, $\therefore \cap \overline{\mathcal{M}} \neq \phi$

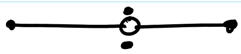
Previously, we proved

$A \subset X$ is closed $\xRightarrow{\text{given } X \text{ is compact}}$ A is compact

Qu. Is it true that

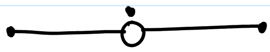
$A \subset X$ is compact $\xRightarrow{X \text{ is compact}}$ A is closed.

Example.

$[-1,1] \sqcup [-1,1] \xrightarrow{q}$ 

compact ∴ compact

$[-1,1]$ $A = q([-1,1])$



Is it closed?

Theorem Let X be Hausdorff.

If $A \subset X$ is compact then A is closed.

Corollary X is cpt T_2. $A \subset X$ cpt $\Leftrightarrow$ A is closed.

Theorem. Let X be compact, Y be Hausdorff

A continuous bijection $f: X \longrightarrow Y$ is homeomorphic.

Need to show f^{-1} is continuous

$$\begin{array}{ccc} F \text{ closed in } X & \xleftarrow{f^{-1}} & (f^{-1})^{-1}(F) \text{ closed?} \\ \Downarrow & & \| \quad \Uparrow \\ F \text{ compact} & \Longrightarrow & f(F) \text{ compact} \end{array}$$

Let $A \subset X$ be compact and X be Hausdorff

Need to show $A \supset \overline{A}$ or $X \setminus A \in \mathcal{T}$

Take any $x \in X \setminus A$

$\vdots$

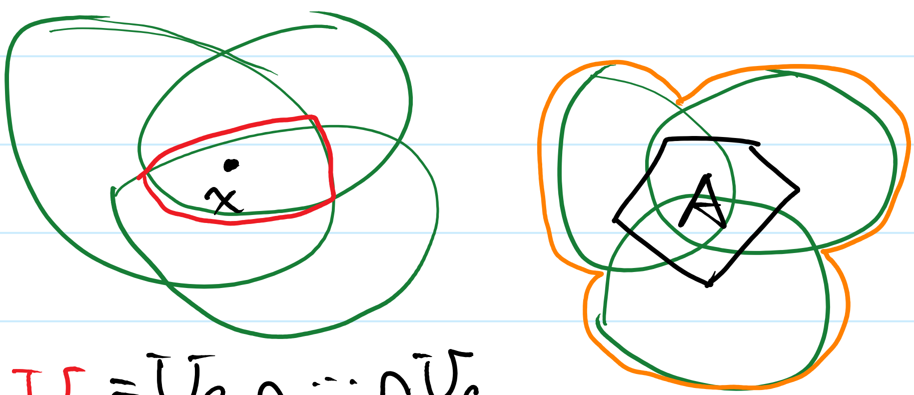
$\exists\ U \in \mathcal{T}$ such that $x \in U \subset X \setminus A$

For each $a \in A$, $x \neq a$

$\exists\ U_a, V_a \in \mathcal{T}$, $x \in U_a$, $a \in V_a$, $U_a \cap V_a = \emptyset$

Then $\mathcal{C} = \{V_a : a \in A\}$ satisfies $\bigcup \mathcal{C} \supset A$

we have $V_{a_1} \cup V_{a_2} \cup \cdots \cup V_{a_n} \supset A$



$U = U_{a_1} \cap \cdots \cap U_{a_n}$

$V = V_{a_1} \cup \cdots \cup V_{a_n}$

Clearly, $x \in U \subset X \setminus V \subset X \setminus A$ $\square$

Actually proved

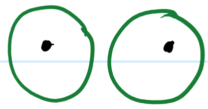
$\forall$ compact $A \subset X$ and $x \notin A$

$\exists\ U, V \in \mathcal{T}$ $x \in U$, $A \subset V$, $U \cap V = \emptyset$

Look familiar?

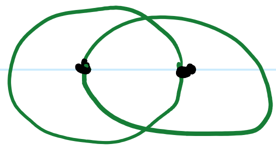
Separation Axioms on $(X, \mathcal{T})$

Hausdorff (T_2): $\forall\ x \neq y\ \exists\ U, V \in \mathcal{T}$ such that $x \in U,\ y \in V,\ U \cap V = \phi$



$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

T_1: $\forall\ x \neq y\ \exists\ U, V \in \mathcal{T}$ such that $x \in U \setminus V,\ y \in V \setminus U$

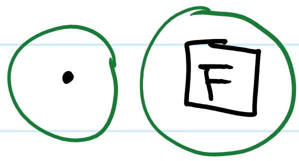


Fact. $T_1 \Leftrightarrow$ singleton is closed

$x \neq y \Leftrightarrow x \in X \setminus \{y\}$

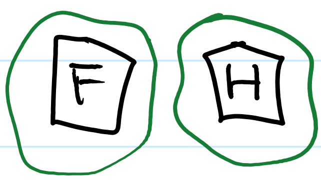
$y \notin U,\ x \in U \Leftrightarrow x \in U \subset X \setminus \{y\}$

Regular: $\forall\ x \notin$ closed F, $\exists\ U, V \in \mathcal{T}$ such that $x \in U,\ F \subset V,\ U \cap V = \phi$



T_3 : T_1 + regular

Normal: $\forall$ closed F, H with $F \cap H = \phi$ $\exists\ U, V \in \mathcal{T}$ such that $F \subset U,\ H \subset V,\ U \cap V = \phi$



T_4 : T_1 + normal

Deep Theorem (Urysohn Lemma)

Let $A, B \subset X$ be closed and X be normal.

Then $\exists$ continuous $f: X \longrightarrow [0,1]$ such that

$$f|_A \equiv 0, \quad f|_B \equiv 1.$$

Tietz Extension true for normal spaces.

Good about regularity

Let $x \in U$ where $U \in \mathcal{J}$

Then $X \setminus U$ is closed and $x \notin X \setminus U$

By regularity, we have $U_1, V \in \mathcal{J}$ such that

$$\underbrace{x \in U_1, \quad X \setminus U \subset V, \quad U_1 \cap V = \emptyset}$$

$$x \in U_1 \subset \underset{\text{closed}}{X \setminus V} \subset U$$

$$\therefore \quad x \in U_1 \subset \overline{U_1} \subset U$$

Thus, we have

$$x \in \cdots\cdots U_n \subset \overline{U_n} \subset U_{n-1} \subset \cdots U_1 \subset \overline{U_1} \subset U$$

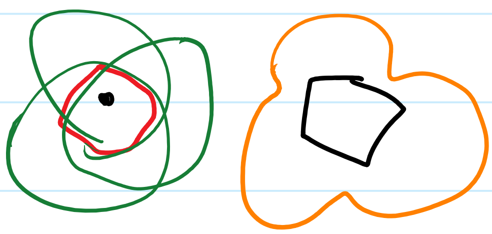
T_3 : T_1 + regular and so T_2

In T_3 space, a closed subset is compact

Then $x \in U_1 \subset K \subset U$, where K is compact

Compact Hausdorff.

From the proof,



X is actually regular, with given T_2, $\therefore$ T_3

Do the proof again for closed F, H, $F \cap H = \emptyset$

X is also normal, $\therefore$ T_4